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Value distribution and spectral analysis of differential operators

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Abstract. The theory of value distribution is applied, through a study of boundary value distribution for the Weyl–Titchmarsh m-function, to the spectral analysis of differential operators on the half-line. Particular consequences are the theoretical underpinning of a numerical approach to spectral analysis of differential operators exhibiting absolutely continuous or singular spectra, and new identities for the spectral measure in terms of solutions of the associated differential equation.

1. Introduction

This paper is a sequel to [1], in which the theory of value distribution was developed for the class of functions defined by boundary values of Herglotz functions. Recall that the *value distribution* of a real-valued function $F(\lambda)$ is measured by such quantities as $|F^{-1}(S)|$ and $|F^{-1}(S)\cap A|$, where $|\cdot|$ denotes Lebesgue measure, S and A are Borel subsets of $\mathbb{R}$, and where by $F^{-1}(S)$ we mean

$$F^{-1}(S)=\{\lambda\in\mathbb{R};\ F(\lambda)\in S\}.$$

The aim of the present paper is to apply the ideas and methods of [1] to the spectral analysis of differential operators $T=-(\mathrm{d}^2/\mathrm{d}x^2)+V(x)$ in $L^2(0,\infty)$, where $V(\cdot)$ is locally integrable and real valued, but is otherwise unrestricted in its behaviour for large x.

The self-adjoint operator T_α, where α parametrizes the boundary condition at $x=0$, may have discrete, absolutely continuous, or singular continuous spectrum, or a combination of all three. A basic tool of spectral analysis for such differential operators is a study of the boundary behaviour near the real axis of the Weyl–Titchmarsh m-function $m_\alpha(z)$. The m-function is analytic with positive imaginary part in the upper half plane, and its boundary values are defined Lebesgue almost everywhere for $\lambda\in\mathbb{R}$ by $m_\alpha^+(\lambda)=\lim_{\varepsilon\to0^+}m_\alpha(\lambda+\mathrm{i}\varepsilon)$. We are concerned with value distribution for the function $m_\alpha^+(\lambda)$, and its relation to spectral properties of the associated measure μ_α. The behaviour of $m_\alpha^+(\lambda)$ as a function of λ may be very wild indeed. For example, in the presence of singular continuous spectrum $m_\alpha^+(\lambda)$ may assume every real value in every subinterval of $\mathbb{R}$. Nevertheless, the value distribution for $m_\alpha^+(\lambda)$ may be quite regular, and is a key to an understanding of the spectral properties of μ_α.

Two by-products of this work may be mentioned, each of which has implications for the spectral analysis of differential operators. The first is the theoretical framework

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for a method of numerical spectral analysis which is sufficiently general to cover absolutely continuous or singular continuous spectrum. The idea behind this method may be simply stated. It is to solve numerically the associated (real) differential equation, starting at $x = N$ with prescribed 'initial' conditions, and to determine f'/f at $x = 0$, for a sampled range of λ values. The large N distribution of computed values of f'/f is then directly related to spectral behaviour of the differential operator. In particular, if the spectrum is absolutely continuous, local sampling techniques permit the numerical determination of $m_0^+(\lambda)$, despite the fact that this function is complex-valued, and the numerical problems presented by the solution of the differential equation in the complex domain are avoided.

The second by-product of the results of this paper is the derivation of explicit formulae (equation (24) of section 4) for the spectral measure μ_α as the limit of a sequence of absolutely continuous measures, each of which depends in a simple way on the solution at a single point $x = N$ of the real λ differential equation. Recalling that no special conditions have been placed on the large x behaviour of the potential function $V(x)$, these formulae are new, in the degree of generality presented here.

The organization of the paper is as follows. In section 2, we introduce the family of differential operators $\{T_\alpha\}$ with their associated m-functions and measures μ_α. Most of section 3 is based on theorem 1 and its corollary, which establish the basic formulae (equations (12) and (15)) for value distribution of $m_0^+(\lambda)$. Through this theorem, a link is forged between the family of spectral measures $\{\mu_\alpha\}$ and a Cauchy distribution function $\omega(t, S)$ which controls local value distribution. Theorem 1 is next applied to the Weyl–Titchmarsh m-functions over a *finite* interval in order to relate local spectral properties to sampled value distribution for f'/f at $x = 0$.

Finally, in section 4, asymptotic value distribution for f'/f at large N is discussed, leading in theorem 2 to the characterization of μ_α as a weak limit of absolutely continuous measures.

2. Spectral analysis and the Weyl–Titchmarsh *m*-function

We consider the family of differential operators $\{T_\alpha\}$, $-\pi/2 < \alpha < \pi/2$, acting in $L^2(\mathbb{R}_+)$, where T_α is defined by

$$T_\alpha = -\frac{d^2}{dx^2} + V(x)$$

subject to the boundary condition

$$(\cos\alpha) f(0) + (\sin\alpha) f'(0) = 0 \tag{1}$$

and $V(\cdot)$ is a real-valued function such that $V \in L_1(0, N)$, for any $N > 0$.

The operator T_α is defined in the first instance on the set of infinitely differentiable functions having compact support in $[0, \infty)$, and for which the boundary condition (1) is satisfied. (By derivatives at $x = 0$ we mean right derivatives.) Assuming the limit-point case ([2], [3]) at infinity, T_α is essentially self-adjoint on this set of functions, and therefore extends to a self-adjoint operator in $L^2(\mathbb{R}_+)$.

Associated with the differential expression $-(d^2/dx^2) + V(x)$ is the corresponding Sturm–Liouville differential equation

$$-\frac{d^2}{dx^2} f(x, z) + V(x) f(x, z) = z f(x, z) \qquad (0 < x < \infty, \operatorname{Im} z > 0) \tag{2}$$

with its real counterpart

$$-\frac{d^2}{dx^2}f(x,\lambda)+V(x)f(x,\lambda)=\lambda f(x,\lambda) \qquad (\lambda\in\mathbb{R}). \tag{2'}$$

We denote by $u_\alpha(\cdot,z)$, $v_\alpha(\cdot,z)$ respectively the solutions of equation (2) (with corresponding solutions $u_\alpha(\cdot,\lambda)$, $v_\alpha(\cdot,\lambda)$ of (2′)), subject to the conditions

$$\begin{aligned} u_\alpha(0,z)&=\cos\alpha & v_\alpha(0,z)&=-\sin\alpha \\ u'_\alpha(0,z)&=\sin\alpha & v'_\alpha(0,z)&=\cos\alpha. \end{aligned} \tag{3}$$

The Wronskian of any two solutions of equation (2), for the same value of z, is independent of x, so that we have then

$$W(u_\alpha,v_\alpha)=u_\alpha v'_\alpha-v_\alpha u'_\alpha=1.$$

For fixed $x\geqslant 0$, $u_\alpha(x,z)$ and $v_\alpha(x,z)$ are analytic functions of z, having $u_\alpha(x,\lambda)$ and $v_\alpha(x,\lambda)$ as their values on the real z axis.

The Weyl–Titchmarsh m-function $m_\alpha(z)$ is defined for Im $z>0$ by the condition that $u_\alpha(\cdot,z)+m_\alpha(z)v_\alpha(\cdot,z)\in L^2(0,\infty)$. (See [3]–[6].)

Then $m_\alpha(z)$ is analytic in the upper half-plane, with Im $m_\alpha(z)>0$. The dependence of m_α on α is rather simple. If $f(\cdot,z)$ is any (non-trivial) L^2 solution of equation (2), then

$$\frac{f'(0,z)}{f(0,z)}=\frac{u'_\alpha(0,z)+m_\alpha(z)v'_\alpha(0,z)}{u_\alpha(0,z)+m_\alpha(z)v_\alpha(0,z)}=\frac{\sin\alpha+m_\alpha(z)\cos\alpha}{\cos\alpha-m_\alpha(z)\sin\alpha}$$

so that

$$m_\alpha(z)=\frac{\cos\alpha\, f'(0,z)-\sin\alpha\, f(0,z)}{\sin\alpha\, f'(0,z)+\cos\alpha\, f(0,z)}.$$

Substituting $f'(0,z)/f(0,z)=m_0(z)$, we have

$$m_\alpha(z)=\frac{m_0(z)\cos\alpha-\sin\alpha}{m_0(z)\sin\alpha+\cos\alpha}. \tag{4}$$

It is known, from asymptotic estimates of the solution of the differential equation ([6]), that $\lim_{s\to\infty}|m_0(is)|=\infty$.

Hence equation (4) implies that $\lim_{s\to\infty} m_\alpha(is)=\cot\alpha$ for $\alpha\neq 0$.

Further estimates lead to the conclusion that

$$\int_1^\infty \frac{\text{Im } m_\alpha(is)}{s}<\infty \qquad \text{for any } \alpha\neq 0.$$

The Herglotz representation theorem then allows us to write

$$m_\alpha(z)=\cot\alpha+\int_{-\infty}^{\infty}\frac{d\rho_\alpha(t)}{t-z} \qquad (\alpha\neq 0, \text{Im } z>0) \tag{5}$$

whereas for $\alpha=0$ we have the modified representation

$$m_0(z)=c_1+\int_{-\infty}^{\infty}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)d\rho(t) \tag{5'}$$

with $c_1=\text{Im } m_0(i)$ a real constant.

In equations (5) and (5′), the 'spectral function' ρ_α is uniquely defined, up to an additive constant, and is non-decreasing and right continuous. The importance of ρ_α for spectral analysis lies in the fact that the differential operator T_α is unitarily equivalent to the multiplication operator $(f(t) \to tf(t))$ in the space $L^2(\mathbb{R}, \mathrm{d}\rho_\alpha)$. The spectral properties of T_α are determined, through the measure $\mu_\alpha = \mathrm{d}\rho_\alpha$, by the boundary behaviour of $m_\alpha(z)$ as z approaches the real axis. (See, for example, [3].)

The differential operator T_α may in some sense be regarded as a limit, as $N \to \infty$, of the corresponding operator in the space $L^2(0, N)$. (Indeed, this limiting process may be made precise using the notion of strong resolvent convergence.) It is with this in mind that we introduce m-functions for a *finite* interval $0 \leqslant x \leqslant N$.

To do so, we need to use boundary conditions at $x = N$. For $\mathrm{Im}\, z > 0$, define $m^N_{\alpha,\beta}(z)$ by the condition that $u_\alpha(\cdot, z) + m^N_{\alpha,\beta}(z) v_\alpha(\cdot, z)$ satisfy

$$(\cos\beta) f(N) + (\sin\beta) f'(N) = 0. \tag{6}$$

Hence, for example, $m^N_{0,\beta}(z) = f'(0, z)/f(0, z)$ for the solution $f(\cdot, z)$ of equation (2), subject to

$$f(N, z) = -\sin\beta \qquad f'(N, z) = \cos\beta$$

and the analogous result to (4) becomes

$$m^N_{\alpha,\beta}(z) = \frac{m^N_{0,\beta}(z)\cos\alpha - \sin\alpha}{m^N_{0,\beta}(z)\sin\alpha + \cos\alpha} \qquad \left(\frac{-\pi}{2} < \beta \leqslant \frac{\pi}{2}\right). \tag{7}$$

We can also exhibit the dependence of $m^N_{\alpha,\beta}(z)$ on β. Using $W(u_\alpha, v_\alpha) = 1$ at $x = N$, the above solution $f(\cdot, z)$ of equation (2) may be written as

$$\begin{aligned} f(x, z) = (&u_\alpha(N, z)\cos\beta + u'_\alpha(N, z)\sin\beta) v_\alpha(x, z) \\ &- (v_\alpha(N, z)\cos\beta + v'_\alpha(N, z)\sin\beta) u_\alpha(x, z). \end{aligned}$$

Substituting $m^N_{0,\beta}(z) = f'(0, z)/f(0, z)$ into (7), and using the conditions (3) for u_α, v_α and derivatives at $x = 0$, we have

$$m^N_{\alpha,\beta}(z) = -\left(\frac{u_\alpha(N, z)\cos\beta + u'_\alpha(N, z)\sin\beta}{v_\alpha(N, z)\cos\beta + v'_\alpha(N, z)\sin\beta}\right). \tag{7′}$$

From the differential equation (2) satisfied by $v_\alpha(x, z)$ and $v_\alpha(x, \bar{z})$, we have the elementary identity

$$\frac{\mathrm{d}}{\mathrm{d}x} W(v_\alpha(x, z), v_\alpha(x, \bar{z})) = 2\mathrm{i}\, \mathrm{Im}\, z |v_\alpha(x, z)|^2$$

which on integration wrt x from 0 to N gives

$$\begin{aligned} \mathrm{Im}\left(\frac{v'_\alpha(N, z)}{v_\alpha(N, z)}\right) &= -\frac{1}{2\mathrm{i}} \frac{W(v_\alpha(x, z), v_\alpha(x, \bar{z}))_{x=N}}{|v_\alpha(N, z)|^2} \\ &= -\frac{\mathrm{Im}\, z}{|v_\alpha(N, z)|^2} \int_0^N |v_\alpha(x, z)|^2\, \mathrm{d}x < 0. \end{aligned} \tag{8}$$

Hence $v'_\alpha(N,z)/v_\alpha(N,z)$, and similarly $u'_\alpha(N,z)/u_\alpha(N,z)$, both have negative imaginary part in (7′). From the analytic properties, as a function of z, or u_α and v_α, it follows that $m^N_{\alpha,\beta}(z)$ is analytic in the upper half plane. Integrating the Wronskian identity for the functions

$$u_\alpha(x,z)+m^N_{\alpha,\beta}(z)v_\alpha(x,z) \qquad \text{and} \qquad u_\alpha(x,\bar z)+m^N_{\alpha,\beta}(\bar z)v_\alpha(x,\bar z)$$

shows that $\operatorname{Im} m^N_{\alpha,\beta}(z)>0$; for each boundary condition at $x=N$, parametrized by β, we may regard $m^N_{\alpha,\beta}$ as an equivalent m-function, over a finite interval, to the function $m_\alpha(z)$ for the infinite interval. Indeed, it is an important consequence of the Weyl limit point/limit circle theory ([2], [3]) that $m^N_{\alpha,\beta}(z)$ converges to $m_\alpha(z)$ in the limit as $N\to\infty$, and does so uniformly both in β and in compact subsets of the upper half plane $\mathbb{C}_+$.

As in (5) and (5′), we have the Herglotz representations

$$m^N_{\alpha,\beta}(z)=\cot\alpha+\int_{-\infty}^{\infty}\frac{\mathrm{d}\rho^N_{\alpha,\beta}(t)}{t-z} \qquad (\alpha\neq 0,\ \operatorname{Im} z>0) \tag{9}$$

$$m^N_{0,\beta}(z)=C^N_1(\beta)+\int_{-\infty}^{\infty}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)\mathrm{d}\rho^N_{0,\beta}(t) \tag{9′}$$

and we denote by $\mu^N_{\alpha,\beta}$ the measures $\mathrm{d}\rho^N_{\alpha,\beta}$. The differential operator $T^N_{\alpha,\beta}=-(\mathrm{d}^2/\mathrm{d}x^2)+V(x)$, acting in $L^2(0,N)$ with boundary conditions, parametrized by α and β, at $x=0$ and at $x=N$ respectively, is unitarily equivalent to the multiplication operator in $L^2(\mathbb{R},\mathrm{d}\rho^N_{\alpha,\beta})$.

The decomposition of the measure μ_α into its singular and absolutely continuous parts, $\mu_\alpha=\mu^{\mathrm{s}}_\alpha+\mu^{\mathrm{ac}}_\alpha$, and of μ^{s}_α into its discrete and singular continuous parts μ^{d}_α, μ^{sc}_α, is determined by the boundary behaviour of $m_\alpha(z)$ as z approaches the real axis. We recall, from [1],

$$\mu^{\mathrm{s}}_\alpha=\mu_\alpha\restriction\left\{\lambda\in\mathbb{R}:\ \lim_{\varepsilon\to 0^+}\operatorname{Im} m_\alpha(\lambda+\mathrm{i}\varepsilon)=\infty\right\}$$

$$\mu^{\mathrm{ac}}_\alpha=\mu_\alpha\restriction\{\lambda\in\mathbb{R}:\ \operatorname{Im} m_\alpha(\lambda+\mathrm{i}\varepsilon)\text{ is bounded in }0<\varepsilon\leqslant 1\}$$

$$\mu^{\mathrm{d}}_\alpha=\mu_\alpha\restriction\left\{\lambda\in\mathbb{R}:\ \lim_{\varepsilon\to 0^+}-\mathrm{i}\varepsilon m_\alpha(\lambda+\mathrm{i}\varepsilon)\equiv\mu_\alpha\{\lambda\}\neq 0\right\}.$$

We shall denote by $m^+_\alpha(\lambda)$ the boundary value of $m_\alpha(z)$ for each $\lambda\in\mathbb{R}$ for which this limit exists (finitely); i.e.

$$m^+_\alpha(\lambda)=\lim_{\varepsilon\to 0^+} m_\alpha(\lambda+\mathrm{i}\varepsilon). \tag{10}$$

Then $m^+_\alpha(\lambda)$ is defined for (Lebesgue) almost all $\lambda\in\mathbb{R}$, and $1/\pi\operatorname{Im} m^+_\alpha(\lambda)$ is the density function for μ^{ac}_α. Note from equation (4) that μ^{s}_α, for $\alpha\neq 0$, is supported on the set of $\lambda\in\mathbb{R}$ for which $m^+_0(\lambda)$ exists with $m^+_0(\lambda)=-\cot\alpha$. On the other hand, a set of $\lambda\in\mathbb{R}$ for which $m^+_0(\lambda)$ exists as a *real* limit will have zero μ_0 measure, and zero μ_α measure unless there are points in the set for which $m^+_0(\lambda)=-\cot\alpha$.

A similar analysis applies to the boundary values of $m^N_{\alpha,\beta}(z)$. Moreover, $T^N_{\alpha,\beta}$, being a Sturm–Liouville operator over a finite interval with regular endpoints, has purely discrete spectrum. It follows that each of the spectral measures $\mu^N_{\alpha,\beta}$ is purely discrete, and that $m^N_{\alpha,\beta}(z)$ is meromorphic as a function of z in the entire complex plane. Where no confusion can arise, we shall denote simply by $m^N_{\alpha,\beta}(\lambda)$ the restriction of $m^N_{\alpha,\beta}(z)$ to $z=\lambda$ on the real axis such that λ is not a discrete point of $\mu^N_{\alpha,\beta}$. Thus, for almost all $\lambda \in \mathbb{R}$, $m^N_{\alpha,\beta}(\lambda)$ is the real boundary value of $m^N_{\alpha,\beta}(z)$ as z approaches λ from the upper (or lower) half-plane.

There is, of course, a close connection between spectral properties of T_α and behaviour for large x of solutions of the differential equations (2) and (2′). For example, the discrete points of μ_α consist of those $\lambda \in \mathbb{R}$ for which $\int_0^\infty |v_\alpha(x,\lambda)|^2\,dx < \infty$. Another such link is provided by the notion of subordinacy ([7]–[9]). Thus, $m_0^+(\lambda)$ will exist as a real limit if and only if a value of $\alpha \neq 0$ exists for which the solution $v_\alpha(\cdot,\lambda)$ is subordinate, in which case $m_0^+(\lambda) = -\cot\alpha$. It follows that the singular part of μ_α is concentrated on those $\lambda \in \mathbb{R}$ for which $v_\alpha(\cdot,\lambda)$ is subordinate. Here the solution $v_\alpha(\cdot,\lambda)$ of equation (2′) is said to be subordinate whenever $\lim_{N\to\infty} \int_0^N |v_\alpha(x,\lambda)|^2\,dx / \int_0^N |f(x,\lambda)|^2\,dx = 0$ for any solution $f(\cdot,\lambda)$ of (2′) which is not a constant multiple of $v_\alpha(\cdot,\lambda)$. One of the main results of this paper will be to establish a formula (equation (24)) for the spectral measure μ_α as a limiting integral in terms of the solution $v_\alpha(\cdot,\lambda)$ and its derivative, $v'_\alpha(\cdot,\lambda)$. We shall obtain this result as an application of the theory of value distribution ([1]) for boundary values of Herglotz functions. To apply the ideas and methods of [1], we shall consider first the boundary value distribution of the m-function $m_0(z)$ and its analogue $m^N_{0,\beta}(z)$ for a finite interval, and secondly (in section 4) of the Herglotz function $-v'_\alpha(N,\lambda)/v_\alpha(N,\lambda)$ in the limit $N\to\infty$. It will be seen that value distribution is a key notion in the spectral analysis for differential operators, and in addition provides a practical tool for the investigation of spectral behaviour. Rather than use the results of [1] directly, we shall adapt the treatment to the current context in which the key role is played by the spectral measures μ_α and $\mu^N_{\alpha,\beta}$. For further details, consult [1].

3. Boundary value distribution for $m_\alpha(z)$ and $m^N_{\alpha,\beta}(z)$

For $z \in \mathbb{C}_+$, define a corresponding Cauchy measure $|\cdot|_z$ by

$$|A|_z = \frac{1}{\pi}\int_A \frac{(\operatorname{Im} z)\,dt}{|t-z|^2}$$

where $A \subseteq \mathbb{R}$ is an arbitrary Borel set. For any Borel subset S of $\mathbb{R}$, define

$$\omega(\lambda, S) = \lim_{\varepsilon\to 0^+} |S|_{m_0(\lambda+i\varepsilon)}. \tag{11}$$

Then $\pi\omega(\lambda,S)$ may be interpreted geometrically as the limiting value, as z approaches λ from the upper half plane, of the angle subtended at the point $m_0(z)$ by the subset S of the real axis. As in [1], one may use subsequent results to verify that, for almost all λ having a *real* boundary value for $m_0(z)$,

$$\omega(\lambda,S) = \begin{cases} 1 & \text{for } m_0^+(\lambda) \in S \\ 0 & \text{for } m_0^+(\lambda) \notin S. \end{cases}$$

For such λ, $\omega(\lambda,S)$ may thus be identified almost everywhere with the characteristic function of $(m_0^+)^{-1}(S)$.

The following theorem exhibits the relationship between $\omega(\lambda,S)$ and the family of spectral measures μ_α. As in [1], our results are a development of the ideas and methods of [10].

Theorem 1. Let A, S be arbitrary Borel subsets of $\mathbb{R}$. Then

$$\int_S (1+y^2)^{-1}\mu_{-\cot^{-1}y}(A)\,\mathrm{d}y = \int_A \omega(t, S)\,\mathrm{d}t. \tag{12}$$

Proof. Consider first the case $A=(a, b)$, $S=(c, \infty)$, for $a, b \in \mathbb{R}$ and $c>0$. Equation (12) becomes in that case

$$\int_c^\infty (1+y^2)^{-1}\mu_{-\cot^{-1}y}(a, b)\,\mathrm{d}y = \int_a^b \omega(t, (c, \infty))\,\mathrm{d}t. \tag{12$'$}$$

Define the Herglotz function $G(z)$ ($\operatorname{Im} z>0$) by

$$G(z)=\int_c^\infty \frac{ym_0(z)+1}{y-m_0(z)}\frac{1}{(1+y^2)}\,\mathrm{d}y$$

giving

$$G(z)=\mathrm{i}\pi-\log\left(\frac{m_0(z)-c}{(1+c^2)^{1/2}}\right) \tag{13}$$

where we are defining the log function by

$$\log(r\,\mathrm{e}^{\mathrm{i}\theta})=\log r+\mathrm{i}\theta \qquad \text{for } r>0 \quad \text{and} \quad 0\leqslant\theta<\pi.$$

Note $0\leqslant \operatorname{Im} G(z)\leqslant \pi$ for $\operatorname{Im} z>0$. Hence $G(z)$ has a Herglotz representation of type (5$'$), for which the corresponding measure η is absolutely continuous, with density function given by

$$\xi(\lambda)=\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\operatorname{Im}\left\{\mathrm{i}\pi-\log\left(\frac{m_0(\lambda+\mathrm{i}\varepsilon)-c}{(1+c^2)^{1/2}}\right)\right\}. \tag{13$'$}$$

It is easily verified that $\xi(\lambda)=\omega(\lambda, (c, \infty))$. Hence, with $S=(c, \infty)$, the RHS of equation (12$'$) is just $\eta\{(a, b)\}$. We also have

$$\eta\{(a, b)\}=\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b \operatorname{Im} G(\lambda+\mathrm{i}\varepsilon)\,\mathrm{d}\lambda$$

which on using equation (13) together with equations (4) and (5) in the case $\alpha=-\cot^{-1} y$ leads to

$$\eta\{(a, b)\}=\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_a^b \mathrm{d}\lambda\int_c^\infty \frac{\mathrm{d}y}{(1+y^2)}\left\{\int_{-\infty}^\infty \frac{\varepsilon\,\mathrm{d}\rho_{-\cot^{-1}y}(t)}{(t-\lambda)^2+\varepsilon^2}\right\}. \tag{14}$$

Noting that

$$\operatorname{Im} m_\alpha(i)=\int_{-\infty}^\infty \frac{\mathrm{d}\rho_\alpha(t)}{(t^2+1)}$$

is bounded uniformly in α, by equation (4), it may be verified that the contribution to $\eta\{(a, b)\}$ coming from the t integration over the region $\mathbb{R}\backslash\{(a-\delta, a+\delta)\}$ is of order ε in the limit $\varepsilon\to 0^+$, for any fixed $\delta>0$. Hence the t integral in (14) is effectively over a finite closed interval containing (a, b). The triple integral is then absolutely convergent. Taking first the λ integral and then using the Lebesgue dominated convergence theorem to evaluate the ε limit, we obtain the LHS of (12$'$). Hence equation (12) is proved in the case $A=(a, b)$, $S=(c, \infty)$. A similar proof applies if we replace the semi-infinite interval (c, ∞) by $(-\infty, -c)$, and using countable additivity the theorem follows for general Borel sets A and S.

Corollary 1. For any Borel set A, let

$$A_0=A\cap\{\lambda\in\mathbb{R}: \lambda\in \operatorname{dom}(m_0^+) \text{ and } \operatorname{Im} m_0^+(\lambda)=0\}.$$

Then, for Borel subsets A, S of $\mathbb{R}$,

$$|(m_0^+)^{-1}(S)\cap A| = \int_S (1+y^2)^{-1}\mu_{-\cot^{-1}y}(A_0)\,\mathrm{d}y. \tag{15}$$

This corollary implies the remarkable result that the density function for the value distribution of $m_0^+(\lambda)$, where m_0^+ is real, over a set A, is a simple function of the spectral measures μ_α for the one-parameter family T_α of self-adjoint realizations of the differential operator $-(\mathrm{d}^2/\mathrm{d}x^2)+V(x)$.

Theorem 1 and its corollary apply also to the functions $m_{\alpha,\beta}^N(z)$. Indeed, $m_{\alpha,\beta}^N(z)$ has a *real* boundary value as $z\to\lambda$, for almost all $\lambda\in\mathbb{R}$. With m_0 replaced by $m_{0,\beta}^N$ in equation (13′), the density function $\xi_\beta^N(\lambda)$ may be identified almost everywhere with the characteristic function of $(m_{\alpha,\beta}^N)^{-1}(S)$, and we have

Corollary 2. For arbitrary Borel subsets A, S of $\mathbb{R}$,

$$\int_S (1+y^2)^{-1}\mu^N_{-\cot^{-1}y,\beta}(A)\,\mathrm{d}y = |(m_{0,\beta}^N)^{-1}(S)\cap A|. \tag{16}$$

Equations (12) and (16) together imply the existence of an *asymptotic* value distribution for $m_{0,\beta}^N(\lambda)$, in the limit as $N\to\infty$. This result may be summarized in the following corollary.

Corollary 3. For arbitrary Borel subsets A, S of $\mathbb{R}$,

$$\lim_{N\to\infty}|(m_{0,\beta}^N)^{-1}(S)\cap A| = \int_A \omega(t,S)\,\mathrm{d}t. \tag{17}$$

Proof. We consider first the case in which A is a finite open interval, $A=(a,b)$. The Weyl limit point/limit circle theory implies that $m_{0,\beta}^N(i)$ lies on a circle C_i^N in the upper half plane, such that for $N'>N$ the circle $C_i^{N'}$ lies in the interior of the circle C_i^N. Hence, we have bounds $|m_{0,\beta}^N(i)|\leqslant$ const and $\operatorname{Im} m_{0,\beta}^N(i)\geqslant \text{const}>0$, uniformly in N for $N\geqslant 1$, say, and for β in the range $-\pi/2<\beta\leqslant\pi/2$. Using these bounds in the numerator and denominator of (7), they can be extended to apply also to $m_{\alpha,\beta}^N(i)$, giving in particular $|m_{\alpha,\beta}^N(i)|\leqslant$ const, uniformly in α, β, and in N for $N\geqslant 1$.

From the imaginary parts of equations (9) and (9′), this gives

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\rho_{\alpha,\beta}^N(t)}{(t^2+1)}\leqslant \text{const}$$

from which it follows, for fixed a, b, that $\mu_{\alpha,\beta}^N\{(a,b)\}\leqslant$ const.

We also have $\lim_{N\to\infty}\mu_{\alpha,\beta}^N\{(a,b)\}=\mu_\alpha\{(a,b)\}$, provided a, b are not discrete points of μ_α. However, λ can be a discrete point of μ_α only if $m_0^+(\lambda)=-\cot\alpha$. This can happen at $\lambda=a$, or at $\lambda=b$ only for at most two values of $-\cot\alpha$. With $A=(a,b)$, the Lebesgue dominated-convergence theorem now gives

$$\begin{aligned}\lim_{N\to\infty}&|(m_{0,\beta}^N)^{-1}(S)\cap A|\\ &=\lim_{N\to\infty}\int_S(1+y^2)^{-1}\mu^N_{-\cot^{-1}y,\beta}(A)\,\mathrm{d}y\\ &=\int_S(1+y^2)^{-1}\mu_{-\cot^{-1}y}(A)\,\mathrm{d}y=\int_A\omega(t,S)\,\mathrm{d}t\end{aligned}$$

from equation (12).

The result may be extended to general Borel sets A, on noting that $|(m_{0,\beta}^N)^{-1}(S)\cap A|\leqslant|A|$ and $\omega(t,S)\leqslant 1$, and using countable additivity for Lebesgue measure.

Corollary 3 provides a theoretical justification for the numerical approach to spectral analysis which was sketched in the introduction. Let λ_0 be any value of the spectral parameter λ at which the limit $m_0^+(\lambda)$ exists, as in equation (10). (This limit exists in any case for (Lebesgue) almost all values of λ.) Take a small interval $(\lambda_0-\delta, \lambda_0+\delta)$ containing λ_0. For λ in this interval, solve the differential equation $-(\mathrm{d}^2f/\mathrm{d}x^2)+Vf=\lambda f$ from $x=N$ to $x=0$, starting at $x=N$ with the 'initial' conditions

$$f(N,\lambda)=-\sin\beta \qquad f'(N,\lambda)=\cos\beta$$

where β is arbitrary but fixed, in the range $-\pi/2<\beta\leqslant\pi/2$. At $x=0$, evaluate $f'(0,\lambda)/f(0,\lambda)$ to give $m_{0,\beta}^N(\lambda)$. The distribution of the values of this ratio, for $\lambda\in(\lambda_0-\delta,\lambda_0+\delta)$, in the limit as $N\to\infty$, will reflect the spectral properties of the differential operator $T_0=-(\mathrm{d}^2/\mathrm{d}x^2)+V$ in the neighbourhood of $\lambda=\lambda_0$.

In particular, we may adopt a sampling procedure in which λ is chosen randomly, with uniform distribution in the interval $(\lambda_0-\delta,\lambda_0+\delta)$. In that case, according to corollary 3, the limiting probability distribution for $m_{0,\beta}^N(\lambda)$ for large N is given by

$$\lim_{N\to\infty}\operatorname{prob}\{m_{0,\beta}^N(\lambda)\in S\}=\frac{1}{2\delta}\int_{\lambda_0-\delta}^{\lambda_0+\delta}\omega(t,S)\,\mathrm{d}t. \tag{18}$$

By theorem 3 and equation (20) of [1], the RHS of (18) converges to $\omega(\lambda_0,S)$ in the limit as $\delta\to0^+$. Hence, if the interval containing λ_0 is chosen to be small enough, the limiting probability in (18) will be close to $\omega(\lambda_0,S)$.

Two possibilities arise, each illustrating a different mode of spectral behaviour near λ_0. If $m_0^+(\lambda_0)$ is real then $\omega(\lambda_0,S)=1$ for any open interval S containing $m_0^+(\lambda_0)$. The limiting probability distribution, for small δ, then approaches a point distribution. concentrated with probability 1 at $m_0^+(\lambda_0)$. Numerically, this corresponds to finding $f'(0,\lambda)/f(0,\lambda)$ close to $m_0^+(\lambda_0)$ with higher and higher frequency, the closer the sampled λ values are taken to λ_0. If this happens for almost all λ_0 then we are dealing with a case of purely singular spectrum for T_0.

Alternatively, it may happen that $\operatorname{Im} m_0^+(\lambda_0)\neq0$. Again we have, in this case, a limiting probability distribution close to $\omega(\lambda_0,S)$, if δ is small enough. However, according to equation (11), $\omega(\lambda_0,S)$ now defines a *Cauchy probability distribution*, having density function

$$\frac{1}{\pi}\frac{\operatorname{Im} m_0^+(\lambda_0)}{(\lambda-\operatorname{Re} m_0^+(\lambda_0))^2+(\operatorname{Im} m_0^+(\lambda_0))^2}.$$

Here, if λ is sampled close to λ_0, the computed values of $f'(0,\lambda)/f(0,\lambda)$, rather than accumulating in the proximity of the fixed value $m_0^+(\lambda_0)$ (which is now in any case complex), will have a spread described by a Cauchy distribution, of which the median is $\operatorname{Re} m_0^+(\lambda_0)$, and $\operatorname{Im} m_0^+(\lambda_0)$ is the half-width at half-maximum of the density function. Such a sampling technique will therefore permit the estimation of both real and imaginary parts, at λ_0, of the boundary value of the complex m-function $m_0(z)$, from the observed distribution of sample values for $f'(0,\lambda)/f(0,\lambda)$. We are then dealing with absolutely continuous spectrum for T_0; in fact $1/\pi\operatorname{Im} m_0^+(\lambda_0)$ is just the density function, at λ_0, of the absolutely continuous part of the measure μ_0.

Thus the sampled value distribution of $f'(0,\lambda)/f(0,\lambda)$ near $\lambda=\lambda_0$ enables us to distinguish between various types of spectral behaviour. Points λ_0 at which $m_0^+(\lambda)$ is real belong to the support of the singular spectrum of μ_α, where $m_0^+(\lambda_0)=-\cot\alpha$. For points λ_0 in the support of the singular spectrum of μ_0, we have $\omega(\lambda_0,S)=0$ for all finite intervals S, in which case sampled values of $|f'(0,\lambda)/f(0,\lambda)|$ become very large near λ_0. Alternatively, we may describe those points at which $m_0^+(\lambda)$ is real as giving rise to a subordinate solution $\psi_\alpha(\cdot,\lambda_0)$ of the differential equation, with $\psi_0(\cdot,\lambda_0)$ subordinate for λ_0 in the spectral support of μ_0^s.

It should even, in principle, be possible to use these sampling techniques to distinguish between singular continuous and dense discrete spectrum. In both cases, $m_0^+(\lambda)$ will be real for almost all λ. However, discrete points of μ_α, for some $\alpha\neq 0$, coincide, by an adaptation of theorem 4 of [1], with points at which $m_0^+(\lambda)$ is approximately differentiable. At such points λ_0, the sampled values of $(m_{0,\beta}^N(\lambda_0+h)-m_{0,\beta}^N(\lambda_0-h))/2h$, if h is taken from a random uniform distribution over $[0,\delta]$, will accumulate for small positive δ near the value of $m'_{0,\mathrm{app}}(\lambda_0)$, the approximate derivative of m_0 at λ_0; on the other hand, if there is no such discrete point for any non-zero α, this ratio will be predominantly large for h close to zero. Whether this phenomenon represents a realistic practical approach to characterizing these two kinds of singular spectrum is a question requiring further investigation.

4. Spectral measures, and value distribution for $v'_\alpha(N,\lambda)/v_\alpha(N,\lambda)$

Equation (8) shows that $-v'_\alpha(N,z)/v_\alpha(N,z)$ has positive imaginary part in the upper half plane, and is therefore a Herglotz function. We can therefore calculate a value distribution formula for $v'_\alpha(N,\lambda)/v_\alpha(N,\lambda)$, for $\lambda\in\mathbb{R}$. In analogy with the results of section 3, one might expect this ratio to have an asymptotic value distribution in the limit as $N\to\infty$. However, constructed examples $V(x)$ of potential functions for which the spectrum has a singular continuous component show that, in general, $v'_\alpha(N,\lambda)/v_\alpha(N,\lambda)$ need not have an asymptotic value distribution in general. There are, however, important special cases in which there does exist such an asymptotic distribution. We shall exhibit some of these special cases as a consequence of the theory of value distribution for $v'_\alpha(N,\lambda)/v_\alpha(N,\lambda)$ which we shall develop in this section. A further consequence will be the construction of general formulae for the spectral measure μ_α as a weak limit of a sequence of absolutely continuous measures, for each of which the density function will turn out to be a simple rational function of v_α and v'_α. We begin with a lemma, which we state in sufficient generality to cover all intended applications.

Lemma 1. Let S,A be arbitrary Borel subsets of $\mathbb{R}$, and for $N>0$ define the set $\mathcal{S}_N(S,A)$ by

$$\mathcal{S}_N(S,A)=\{\lambda\in A;\ v'_\alpha(N,\lambda)/v_\alpha(N,\lambda)\in S\}. \tag{19}$$

Let $\psi(t)$ be a continuous, positive valued function of t, and let $z(t)$ be a continuous, complex valued function of t such that $\operatorname{Im} z(t)>0$. Then

$$\int_{\mathcal{S}_N(S,A)}\frac{\operatorname{Im} z(t)\psi(t)\,\mathrm{d}t}{|v'_\alpha(N,t)-z(t)v_\alpha(N,t)|^2}=\int_S \mathrm{d}y\int_A\frac{\operatorname{Im} z(t)\psi(t)\,\mathrm{d}\rho^N_{\alpha,-\cot^{-1}y}(t)}{|y-z(t)|^2}. \tag{20}$$

Proof. As for theorem 1, we proceed from special cases in the direction of increasing generality. Consider first of all the case $z(t)\equiv i$, $\psi(t)\equiv 1$, in which equation (20) becomes

$$\int_{\mathscr{S}_N(S,A)} \frac{\mathrm{d}t}{(v'_\alpha(t,N))^2+(v_\alpha(t,N))^2}=\int_S \mathrm{d}y(1+y^2)^{-1}\mu^N_{\alpha,\cot^{-1}y}(A). \qquad (20')$$

As in the proof of theorem 1, take the case $S=(c,\infty)$ and $A=(a,b)$, with $c>0$. We consider the Herglotz function $K(z)$, defined for $\operatorname{Im} z>0$ by

$$K(z)=-\int_c^\infty \frac{u_\alpha(N,z)y-u'_\alpha(N,z)}{v_\alpha(N,z)y-v'_\alpha(N,z)}\frac{1}{(1+y^2)}\,\mathrm{d}y. \qquad (21)$$

The integrand on the RHS of (21) may be written

$$\frac{y-(u_\alpha(N,z)v_\alpha(N,z)+u'_\alpha(N,z)v'_\alpha(N,z))}{(1+y^2)((v'_\alpha(N,z))^2+(v_\alpha(N,z))^2)}$$
$$-\frac{1}{(v'_\alpha(N,z))^2+(v_\alpha(N,z))^2}\frac{1}{(y-v'_\alpha(N,z)/v_\alpha(N,z))}$$

provided $v'_\alpha(N,z)+\mathrm{i}v_\alpha(N,z)\neq 0$, where we have used $W(u_\alpha, v_\alpha)=1$. Carrying out the integration with respect to y now yields

$$K(z)=\frac{1}{(v'_\alpha(N,z))^2+(v_\alpha(N,z))^2}\log\left(\frac{c-v'_\alpha(N,z)/v_\alpha(N,z)}{(1+c^2)^{1/2}}\right)$$
$$-\frac{u_\alpha(N,z)v_\alpha(N,z)+u'_\alpha(N,z)v'_\alpha(N,z)}{(v'_\alpha(N,z))^2+(v_\alpha(N,z))^2}\cot^{-1}c. \qquad (22)$$

Using equation (3) to express u_α, v_α as linear combinations of u_0, v_0, one may verify that the final expression on the RHS of (22) may be written, for $\alpha\neq 0$,

$$\cot^{-1}c\left(\frac{u_0(N,z)v_0(N,z)+u'_0(N,z)v'_0(N,z)-((v'_0(N,z))^2+(v_0(N,z))^2)\cot\alpha}{(v'_\alpha(N,z))^2+(v_\alpha(N,z))^2}\right)$$
$$+\cot\alpha\cot^{-1}c.$$

The singularity, on the RHS of equation (22), at any value of z for which $v'_\alpha(N,z)+\mathrm{i}v_\alpha(N,z)=0$, is removable, since

$$v'_\alpha/v_\alpha=-\mathrm{i}\Rightarrow(u_\alpha v_\alpha+u'_\alpha v'_\alpha)\cot^{-1}c=\mathrm{i}W(u_\alpha,v_\alpha)\cot^{-1}c=\mathrm{i}\cot^{-1}c=\log\left[\frac{c+\mathrm{i}}{(1+c^2)^{1/2}}\right].$$

An alternative evaluation of $K(z)$ may be carried out using the fact that the integrand on the RHS of (21), using equation (7′), is just $m^N_{\alpha,-\cot^{-1}y}(z)/(1+y^2)$. From the Herglotz representations (9) for $m^N_{\alpha,\beta}(z)$, with the above rewriting of the final expression on the RHS of (22), we have the identity

$$\frac{1}{(v'_\alpha(N,z))^2+(v_\alpha(N,z))^2}\log\left[\frac{c-v'_\alpha(N,z)/v_\alpha(N,z)}{(1+c^2)^{1/2}}\right]$$
$$+\cot^{-1}c\left[\frac{u_0(N,z)v_0(N,z)+u'_0(N,z)v'_0(N,z)-((v'_0(N,z))^2+(v_0(N,z))^2)\cot\alpha}{(v'_\alpha(N,z))^2+(v_\alpha(N,z))^2}\right]$$
$$=\int_c^\infty \mathrm{d}y\left\{(1+y^2)^{-1}\int_{-\infty}^{\infty}\frac{\mathrm{d}\rho^N_{\alpha,-\cot^{-1}y}(t)}{(t-z)}\right\} \qquad (23)$$

with a modified right-hand side, to take account of equation (9′). In the case $\alpha=0$, for which the term $-\cot\alpha\cot^{-1}c$ is also removed from the LHS.

Each side of equation (23), with appropriate modification in the case $\alpha = 0$, is a Herglotz function. For $\alpha \neq 0$, one may use known asymptotic behaviour of u_0, v_0 and their derivatives in the limit $s \to \infty$ with $z = is$ (in fact u_0, v_0, u_0', v_0' diverge in this limit, and u_0' diverges more rapidly, by a factor of order $\sqrt{s}$) to show that each side decays to zero along the imaginary axis.

It is also clear that the LHS of equation (23) has a finite boundary value, with $z = \lambda + i\varepsilon$, in the limit $\varepsilon \to 0^+$, except at values of λ for which $v_\alpha'(N, \lambda)/v_\alpha(N, \lambda) = c$. Due to the analytic properties of $v_\alpha'(N, z)/v_\alpha(N, z)$, such values of λ are necessarily isolated, and can give rise only to logarithmic singularities of $K(z)$ on the real axis. Any discrete part of the Herglotz measure η for $K(z)$ would lead to pole singularities, and a singular continuous measure cannot be concentrated on a set of isolated points. Hence η is purely absolutely continuous. Taking the limit as $\varepsilon \to 0^+$ of $\pi^{-1}\,\mathrm{Im}\,K(\lambda + i\varepsilon)$, we find that the density function for η is given almost everywhere by $\chi(\lambda)/(v_\alpha'(N, \lambda))^2 + (v_\alpha(N, \lambda))^2$, where χ is the characteristic function of the set $\{\lambda \in \mathbb{R};\ v_\alpha'(N, \lambda)/v_\alpha(N, \lambda) \in (c, \infty)\}$. A similar analysis applies to the case $\alpha = 0$. Taking the imaginary part of the RHS of equation (23), we may now proceed to evaluate $\eta\{(a, b)\}$ as in equation (14) of the proof of theorem 1, except that here it is the *second* parameter β of $\mu_{\alpha,\beta}^N$ which is integrated over, rather than α. This leads directly to equation (20′), in the case $S = (c, \infty)$, $A = (a, b)$. A similar argument gives equation (20′) with $S = (-\infty, -c)$, again for $A = (a, b)$. The extension of equation (20′) to arbitrary Borel sets A and S is then straightforward.

The proof of equation (20) in the case $z(t) \equiv z_0$, $\psi(t) \equiv 1$, where now z_0 is any fixed complex number in the upper half-plane, is carried out in a precisely similar manner, replacing $(1 + y^2)^{-1}$ in the integrand of the definition (21) of $K(z)$ by $|y - z_0|^{-2}$. The density function for the measure η then becomes $\chi(\lambda)/|v_\alpha'(N, \lambda) - z_0 v_\alpha(N, \lambda)|^2$, and equation (20) follows as before, including the case $\alpha = 0$.

We have now proved equation (20) in the case that the functions $z(t)$ and $\psi(t)$ are constant over the set A. By linearity, we can also treat the situation in which $z(t)$ and $\psi(t)$ are step functions. The extension to continuous functions now follows standard arguments, and the lemma is now proved in full generality.

As illustrations of the role of equation (20) in describing value distribution for $v_\alpha'(N, \lambda)/v_\alpha(N, \lambda)$ and its behaviour in the large N limit, we consider two applications.

For the first, take $A = (a, b)$ to be a finite interval, of which the endpoints are not discrete points of μ_α. Set $\psi(t) \equiv 1$ and $S = \mathbb{R}$. In the limit $N \to \infty$, $\mu_{\alpha,\beta}^N(I)$ converges to $\mu_\alpha(I)$, again for intervals I of which the endpoints are not discrete points of μ_α. One can then justify taking the limit $N \to \infty$ under the integral sign on the RHS of (20). Having done so, the y integration may be carried out explicitly, and we have the following characterization of the spectral measure μ_α as a weak limit of a sequence of absolutely continuous measures.

Theorem 2. Let (a, b) be an interval for which neither a nor b are discrete points of μ_α. Let $z(t)$ be an arbitrary continuous, complex valued function of t, such that $\mathrm{Im}\, z(t) > 0$. Then

$$\mu_\alpha\{(a, b)\} = \lim_{N \to \infty} \frac{1}{\pi} \int_a^b \frac{\mathrm{Im}\, z(t)\, \mathrm{d}t}{|v_\alpha'(N, t) - z(t) v_\alpha(N, t)|^2}. \tag{24}$$

In particular (with $z(t) \equiv i$) we have

$$\mu_\alpha\{(a, b)\} = \lim_{N \to \infty} \frac{1}{\pi} \int_a^b \frac{\mathrm{d}t}{(v_\alpha'(N, t))^2 + (v_\alpha(N, t))^2}. \tag{24′}$$

A more transparent statement of the relation between the measures μ_α and the asymptotic behaviour of the corresponding solution v_α of the differential equation (2′) could hardly be envisaged! We also have

$$\int_a^b \psi(t)\,\mathrm{d}\rho_\alpha(t) = \lim_{N\to\infty} \frac{1}{\pi}\int_a^b \frac{\psi(t)\,\mathrm{d}t}{(v'_\alpha(N,t))^2+(v_\alpha(N,t))^2}. \tag{25}$$

Equation (24′) shows that if $(v'_\alpha(N,t))^2+(v_\alpha(N,t))^2 \geqslant \text{const} > 0$ then μ_α will be purely absolutely continuous in the interval (a, b) (or, more generally, if $(v'_\alpha(N,t))^2+(v_\alpha(N,t))^2 \geqslant h(t)$, where $1/h \in L_1(a,b)$. The following theorem determines the density function of μ_α and the asymptotic value distribution of $v'_\alpha(N,\lambda)/v_\alpha(N,\lambda)$ in a wide variety of cases for which μ_α is absolutely continuous.

Theorem 3. Suppose there exists a continuous complex valued function $k(t)$, $a \leqslant t \leqslant b$, with $\operatorname{Im} k(t) > 0$, such that the limit

$$R(\lambda) \equiv \lim_{N\to\infty} |v'_\alpha(N,\lambda) - k(\lambda)v_\alpha(N,\lambda)| \tag{26}$$

exists uniformly on (a, b), and is non-zero. Then:
(i) The measure μ_α is absolutely continuous on (a, b), with density function

$$\frac{1}{\pi}\frac{\operatorname{Im} k(\lambda)}{(R(\lambda))^2}.$$

(ii) $v'_\alpha(N,\lambda)/v_\alpha(N,\lambda)$ has an asymptotic value distribution, in the sense that

$$\lim_{N\to\infty} |\{\lambda \in A;\, v'_\alpha(N,\lambda)/v_\alpha(N,\lambda) \in S\}| = \int_A |S|_{k(t)}\,\mathrm{d}t \tag{27}$$

for Borel sets A, S with $A \subseteq (a, b)$.

Proof. Conclusion (i) of the theorem follows immediately from equation (24), applied to subintervals of (a, b) with $z(t) = k(t)$.

To prove (ii) of the theorem, set $\psi(t) = |v'_\alpha(N,t) - k(t)v_\alpha(N,t)|^2/\operatorname{Im} k(t)$ and $z(t) = k(t)$ in equation (20). According to definition (19) of the set $\mathscr{S}_N(S, A)$, the LHS of (20) may then be identified with the Lebesgue measure of the set considered in (19). On the RHS of (20), we have, then,

$$\int_S \mathrm{d}y \int_A \frac{|v'_\alpha(N,t) - k(t)v_\alpha(N,t)|^2\,\mathrm{d}\rho^N_{\alpha,-\cot^{-1}y}(t)}{|y-k(t)|^2}$$

where, without loss of generality, we take A to be a subinterval of (a, b).

In the limit as $N\to\infty$ it is straightforward to show that $|v'_\alpha(N,t) - k(t)v_\alpha(N,t)|^2$ may be replaced by its uniform limit $(R(t))^2$. Since $R(t)$ is continuous we may proceed to the limit $N\to\infty$ to obtain

$$\int_S \mathrm{d}y \int_A \frac{(R(t))^2}{|y-k(t)|^2}\,\mathrm{d}\rho_\alpha(t).$$

However, (i) of the theorem implies

$$\mathrm{d}\rho_\alpha(t) = \frac{1}{\pi}\frac{\operatorname{Im} k(t)}{(R(t))^2}\,\mathrm{d}t$$

from which (ii) of the theorem follows, remembering that $|S|_{k(t)}$ is the Cauchy measure of S, given by

$$|S|_{k(t)}=\frac{1}{\pi}\int_S \frac{\operatorname{Im} k(t)}{|y-k(t)|^2}\,\mathrm{d}y.$$

Remarks

Observe the close similarity between equation (27) of theorem 2 and the corresponding equation (17) giving asymptotic value distribution for $m_{0,\beta}^{N}(\lambda)$, where also $\omega(t,S)$ defines a Cauchy (or point) distribution for almost all t. The correspondence between the two equations becomes even closer if we regard $k(\lambda)$ in equation (26) as the analogue of $m_0^+(\lambda)$.

Theorem 3 applies, with $k(\lambda)=\mathrm{i}\sqrt{\lambda}$, to any potential function $V(x)$ such that $V\in L_1(0,\infty)$, where $[a,b]$ is a subinterval of $(0,\infty)$, as well as to Coulomb-like potentials such as $V(x)=\text{const}/(1+x)$ and to a variety of other long range potentials. Since the proof of theorem 3 requires only that the limit in equation (26) exist for a *subsequence* $\{N_j\}$ such that $\lim_{j\to\infty} N_j=\infty$, the theorem may also be applied to the case of periodic potentials and their perturbations, where the sequence $\{N_j\}$ is related to the periodicity τ of the potential, by taking $N_{j+1}-N_j=\tau$.

Results similar to (ii) of theorem 3 have been obtained for a general class of Sturm–Liouville differential operators by Atkinson [11], and have been extended by Clark to Hamiltonian systems, in [12].

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